The spectrum and the Palais-Smale condition for a self-adjoint operator

C. A. Stuart
Département de Mathématiques,
Ecole Polytechnique Fédérale Lausanne,
CH-1015 Lausanne, Switzerland

1 Introduction

The purpose of this note is to point out a precise and useful relationship between two important notions in operator theory and variational analysis. In the theory of linear operators, the spectrum and its refinements, particularly the essential spectrum, are fundamental concepts. In the study of critical points of real-valued functionals, with or without constraints, the Palais-Smale condition and its variants play an essential role. For a self-adjoint operator, S, the spectrum and essential spectrum of S can be characterized in terms of (P-S) conditions for and the associated quadratic form, J, and its Rayleigh quotent, j, respectively. In the case of a bounded operator the results can be stated very simply.

Consider a bounded self-adjoint operator $S : H \to H$ acting on a real Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and let $J(u) = \langle Su, u \rangle$ for $u \in H$. Let σ and σ_e denote the spectrum and essential spectrum of S, respectively. Set $M = \{u \in H : \langle u, u \rangle = 1\}$ and let j denote the restriction of J to M. Then,

$$\sigma = \{\lambda \in \mathbb{R} : \text{ the functional } J_\lambda \text{ does not satisfy (P-S) on } H\}$$

where $J_\lambda(u) = J(u) - \lambda \langle u, u \rangle$ for $u \in H$ and

$$\sigma_e = \{\lambda \in \mathbb{R} : \text{ the functional } j \text{ does not satisfy (P-S) at level } \lambda \text{ on } M\}.$$

There are analogous results for unbounded operators but their statement requires a little more care since J is not differentiable with respect to the norm of H even at points in the domain of S. However there is a natural domain, H_1, and norm associated with the form J and, once this has been introduced, similar relations hold.

The main definitions are recalled in Section 2. The auxiliary space H_1, required to deal with unbounded operators is introduced in Section 3 where the main step involves showing that S is equivalent to a bounded self-adjoint operator on the Hilbert space H_1. This construction in studied in some detail decause of its use in nonlinear analysis. (See [4] and [5], for example.) Our results give a simpler and more complete description of the relationship betweem the operator $S - \lambda I$ on H and its representation $A - \lambda L$ in H_1 than was previously available. The extensions of the quadratic form for S, and its Rayleigh quotent, are discussed in Section 4. The results relating the spectrum of S the (P-S) conditions are stated and proved in full generality in Section 5. When S is bounded the discussion can be abridged by setting $H_1 = H, A = S$ and $L = I$. Then Section 3 can be ignored and some parts of the proofs in Sections 4 and 5 can be simplified.

2 Preliminaries

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with norm $\|\cdot\|$ and consider a self-adjoint operator, $S : D(S) \subset H \to H$ acting in H where $D(S)$ is a dense subspace of H. The following definitions are standard. See [2],[9],[7] or [10].

$$\rho(S) = \{\lambda \in \mathbb{R} : \text{ the operator } S - \lambda I : D(S) \to \ H \text{ is an isomorphism}\}$$

$$\sigma(S) = \mathbb{R} \backslash \rho(S)$$

$$\sigma_d(S) = \{\lambda \in \sigma(S) : \text{ the operator } S - \lambda I : D(S) \to \ H \text{ is Fredholm}\}$$

$$\sigma_e(S) = \sigma(S) \backslash \sigma_d(S)$$

Here $\rho(S)$ is usually called the resolvent set and $\sigma_d(S)$ the discrete spectrum. The spectrum, $\sigma(S)$, is always a closed non-empty set whereas the essential spectrum, $\sigma_e(S)$, is a closed subset of $\sigma(S)$ which may be empty. Since a self-adjoint operator is always closed, the resolvent set consists of those $\lambda \in \mathbb{R}$ such the $S - \lambda I : D(S) \subset H \to H$ has a bounded inverse defined on all of

H. The discrete spectrum consists of the eigenvalues of S which have finite multiplicity and which are isolated points of $\sigma(S)$. The splitting of $\sigma(S)$ into $\sigma_d(S) \cup \sigma_e(S)$ is also important because $\sigma_e(S)$ is invariant under compact perturbation of S.

For general closed linear operators several different notions of what is meant by the essential spectrum are used. However, in the case of self-adjoint operators, they all coincide with the above definition. See [2] or [7]. This is one reason for restricting our discussion to the self-adjoint case.

Since it was first formulated by Palais and Smale under the name of Condition (C), variants of their idea have become a standard part of variational methods. See [1],[3] or [8]. Consider a smooth $H-$manifold V and a functional $f \in C^1(V, \mathbb{R})$. (In fact we shall only use two trivial cases, namely $V = H$ and $V = \{u \in H : g(u) = 0\}$ where $g \in C^\infty(H, \mathbb{R})$ with $g'(u) \neq 0$ for all $u \in V$.) See [1] §27.4 or Chapter 43 of [8]. For $c \in \mathbb{R}$, the functional f satisfies the Palais-Smale condition, (P-S)$_c$, at the level c on V if every sequence $\{u_n\} \subset V$ such that

$$f(u_n) \to c \text{ and } f'(u_n) \to 0$$

has a subsequence which converges in H. If f satisfies (P-S)$_c$ for every $c \in \mathbb{R}$, then f is said to satisfy the Palais-Smale condition, (P-S), on V. In these definitions $f'(u)$ is a bounded linear functional on the tangent space to V at u, and $f'(u_n) \to 0$ in the sense that $\|f'(u_n)\|_* \to 0$ where $\|\cdot\|_*$ denotes the norm on the dual space $T_u(V)^*$.

The kernel and range of a linear operator T will be denoted by $\ker T$ and $rgeT$, respectively.

Finally we recall a formula concerning Stieltjes integrals which will be used in conjunction with the spectral theorem Section 3. For $-\infty < a < b < \infty$, let f and g be continuous functions on $[a, b]$ and let h be a function of bounded variation on $[a, b]$. Then

$$\int_a^b f(x) d\left[\int_a^x g(t) dh(t)\right] = \int_a^b f(x) g(x) dh(t). \tag{1}$$

3 The form space of a self-adjoint operator

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with norm $\|\cdot\|$ and consider a self-adjoint operator, $S : D(S) \subset H \to H$ acting in H where $D(S)$ is a dense subspace of

H. There is a unique right-continuous resolution of the identity (or spectral family, [10]) $\{E(\lambda) : \lambda \in \mathbb{R}\}$ such that

$$D(S) = \left\{ u \in H : \int \lambda^2 d \langle E(\lambda)u, u \rangle < \infty \right\} \tag{2}$$

and

$$\langle Su, v \rangle = \int \lambda d \langle E(\lambda)u, v \rangle \text{ for all } u \in D(S) \text{ and } v \in H. \tag{3}$$

When no domain of integration is indicated it is understood that the integral is over $\mathbb{R}$. We refer to [6] for the basic results about Steiltjes integrals which we shall use. For any continuous function, $f : \sigma(S) \to \mathbb{R}$, a self-adjoint operator, $f(S) : D(f(S)) \subset H \to H$, is defined by

$$D(f(S)) = \left\{ u \in H : \int f(\lambda)^2 d \langle E(\lambda)u, u \rangle < \infty \right\} \tag{4}$$

and

$$\langle f(S)u, v \rangle = \int f(\lambda) d \langle E(\lambda)u, v \rangle \text{ for all } u \in D(f(S)) \text{ and } v \in H. \tag{5}$$

In particular, $D(f(S))$ is a dense subspace of H. For all $u \in D(f(S))$,

$$\|f(S)u\|^2 = \int f(\lambda)^2 d \langle E(\lambda)u, u \rangle \tag{6}$$

and

$$E(\lambda)u \in D(f(S)) \text{ with } f(S)E(\lambda)u = E(\lambda)f(S)u \text{ for all } \lambda \in \mathbb{R}. \tag{7}$$

The **form space** of $S : D(S) \subset H \to H$ is now defined as the domain of the operator $|S|^{1/2}$ equipped with its graph norm. (See page 183 of [2].) More explicitly, we set

$$H_1 = D(|S|^{1/2})$$

with

$$\langle u, v \rangle_1 = \langle u, v \rangle + \left\langle |S|^{1/2} u, |S|^{1/2} v \right\rangle$$

and

$$\|u\|_1^2 = \|u\|^2 + \left\| |S|^{1/2} u \right\|^2$$

for all $u, v \in H_1$.
It is well-known that $(H_1, \langle\cdot,\cdot\rangle_1)$ is a Hilbert space and that $D(S)$ is a dense subspace of $(H_1, \langle\cdot,\cdot\rangle_1)$. Furthermore,

$$H_1 = \left\{ u \in H : \int |\lambda| \, d\langle E(\lambda)u, u\rangle < \infty \right\} \tag{8}$$

and

$$\|u\|_1^2 = \int (1 + |\lambda|) \, d\langle E(\lambda)u, u\rangle . \tag{9}$$

Lemma 1 *Let $S : D(S) \subset H \to H$ be a self-adjoint operator on H with form space $(H_1, \langle\cdot,\cdot\rangle_1)$. Consider the self-adjoint operator, $T : D(T) \subset H \to H$ on H defined by $T = (I + |S|)^{1/2}$. Then*
(a) $D(T) = H_1$ and T is an isometric isomorphism of the Hilbert space $(H_1, \langle\cdot,\cdot\rangle_1)$ onto $(H, \langle\cdot,\cdot\rangle)$. Also,
(b) $D(S) = \{u \in H_1 : Tu \in H_1\}$ and

$$\|u\|^2 + \|Su\|^2 \leq \|Tu\|_1^2 \leq 2\left(\|u\|^2 + \|Su\|^2\right) \text{ for all } u \in D(S).$$

Finally,
(c) $T^{-1}u \in D(S)$ and

$$ST^{-1}u = T^{-1}Su \text{ for all } u \in D(S).$$

Remark It follows from (a) that

$$\left\langle T^{-1}u, T^{-1}v \right\rangle_1 = \langle u, v\rangle \text{ for all } u, v \in H. \tag{10}$$

Proof (a) By definition,

$$D(T) = \left\{ u \in H : \int (1 + |\lambda|) \, d\langle E(\lambda)u, u\rangle \right\}$$

and so by (8) and (9), $H_1 = D(T)$ with

$$\|Tu\|^2 = \int (1 + |\lambda|) \, d\langle E(\lambda)u, u\rangle = \|u\|_1^2 \text{ for all } u \in H_1$$

by (6) and (9). We have already shown that the operator $T : H_1 \to H$ is isometric. This implies that $rge(T) = \{Tu : u \in H_1\}$ is a closed subset of H. But the self-adjointness of $T : D(T) \subset H \to H$ now yields

$$rge(T) = \ker(T)^{\perp} = \{0\}^{\perp} = H$$

and so $T : H_1 \to H$ is an isometric isomorphism.

(b) Suppose first that $u \in H_2$. To prove that $Tu \in H_1$, we must show that

$$\int (1 + |\lambda|)\, d\langle E(\lambda)Tu, Tu\rangle < \infty.$$

But, by (5), for any $u \in H_1 = D(T)$,

$$\begin{aligned}\langle Tu, E(\lambda)Tu\rangle &= \int (1 + |\mu|)^{1/2}\, d\langle E(\mu)u, E(\lambda)Tu\rangle \\ &= \int_{-\infty}^{\lambda} (1 + |\mu|)^{1/2}\, d\langle E(\mu)u, Tu\rangle\end{aligned}$$

where

$$\begin{aligned}\langle Tu, E(\mu)u\rangle &= \int (1 + |\lambda|)^{1/2}\, d\langle E(\lambda)u, E(\mu)u\rangle \\ &= \int_{-\infty}^{\mu} (1 + |\lambda|)^{1/2}\, d\langle E(\lambda)u, u\rangle .\end{aligned}$$

Thus using the formula in Lemma 5.1(6) on page 221 of [6],

$$\langle Tu, E(\lambda)Tu\rangle = \int_{-\infty}^{\lambda} (1 + |\mu|)\, d\langle E(\mu)u, u\rangle$$

and

$$\int (1 + |\lambda|)\, d\langle E(\lambda)Tu, Tu\rangle = \int (1 + |\lambda|)^2\, d\langle E(\lambda)u, u\rangle$$

for all $u \in H_1$.
Thus for all $u \in H_2, Tu \in H_1$ and

$$\begin{aligned}\|Tu\|_1^2 &= \int (1 + |\lambda|)\, d\langle E(\lambda)Tu, Tu\rangle = \int (1 + |\lambda|)^2\, d\langle E(\lambda)u, u\rangle \\ &\le 2 \int \left(1 + |\lambda|^2\right) d\langle E(\lambda)u, u\rangle = 2\left(\|u\|^2 + \|Su\|^2\right) < \infty.\end{aligned}$$

Conversely, if $u\in H_1$ and $Tu \in H_1$, it follows from (9) that

$$\int (1+|\lambda|)\, d\langle E(\lambda)Tu, Tu\rangle < \infty.$$

But, as above,

$$\int (1+|\lambda|)\, d\langle E(\lambda)Tu, Tu\rangle = \int (1+|\lambda|)^2\, d\langle E(\lambda)u, u\rangle$$

and $(1+|\lambda|)^2 \geq 1+|\lambda|^2$, so that

$$\|Tu\|_1^2 = \int (1+|\lambda|)\, d\langle E(\lambda)Tu, Tu\rangle \geq \int \left(1+|\lambda|^2\right) d\langle E(\lambda)u, u\rangle .$$

Thus, $u \in D(S)$ and

$$\|Tu\|_1^2 \geq \|u\|^2 + \|Su\|^2 .$$

(c) Let $u, v \in H_1$. By (a) and (b), $T^{-1}u \in D(S)$ and

$$\left\langle ST^{-1}u, v\right\rangle = \int \lambda d\left\langle E(\lambda)T^{-1}u, v\right\rangle$$

where

$$\begin{aligned}\left\langle T^{-1}u, E(\lambda)v\right\rangle &= \int (1+|\mu|)^{-1/2}\, d\langle E(\mu)u, E(\lambda)v\rangle \\ &= \int_{-\infty}^{\lambda} (1+|\mu|)^{-1/2}\, d\langle E(\mu)u, v\rangle\end{aligned}$$

and so. The formula in Lemma 5.1(6) on page 221 of [6] now yields

$$\left\langle ST^{-1}u, v\right\rangle = \int \lambda\, (1+|\lambda|)^{-1/2}\, d\langle E(\lambda)u, v\rangle .$$

On the other hand, if $u \in D(S)$ and $v \in H$,

$$\left\langle T^{-1}Su, v\right\rangle = \int (1+|\lambda|)^{-1/2}\, d\langle E(\lambda)Su, v\rangle$$

where

$$\langle Su, E(\lambda)v\rangle = \int \mu d\langle E(\mu)u, E(\lambda)v\rangle = \int_{-\infty}^{\lambda} \mu d\langle E(\mu)u, v\rangle$$

and so now

$$\left\langle T^{-1}Su,v\right\rangle=\int(1+|\lambda|)^{-1/2}\,\lambda d\left\langle E(\lambda)u,v\right\rangle.$$

Hence

$$\left\langle ST^{-1}u,v\right\rangle=\left\langle T^{-1}Su,v\right\rangle\text{ for all }u\in D(S)\text{ and }v\in H_1.$$

Since $H_1=D(T)$ is a dense subspace of H, it follows that $ST^{-1}u=T^{-1}Su$.

Corollary 2 *In the context of the lemma, let $(H_2,\langle\cdot,\cdot\rangle_2)$ denote the graph space of S with*

$$\|u\|_2=\left\{\|u\|^2+\|Su\|^2\right\}^{1/2}\text{ for }u\in H_2.$$

Then
(a) for all $u\in H_2$,

$$\|u\|_2\leq\|Tu\|_1\leq\sqrt{2}\,\|u\|_2\,,\tag{11}$$

(b) $T:H_2\to H_1$ is a homeomorphism, and
(c) $T^{-1}ST^{-1}:H_1\to H_1$ is a bounded linear operator and

$$\left\langle T^{-1}ST^{-1}u,v\right\rangle_1=\langle Su,v\rangle\text{ for all }u\in H_2\text{ and }v\in H_1,$$

(d) $T^{-1}T^{-1}:H\to H_2$ is a bounded linear operator and

$$\left\langle T^{-1}T^{-1}u,v\right\rangle_1=\langle u,v\rangle\text{ for all }u\in H\text{ and }v\in H_1.$$

Proof Part (a) follows immediately from part (b) of Lemma 1. It also shows that T is a bounded operator from $(H_2,\langle\cdot,\cdot\rangle_2)$ into $(H_1,\langle\cdot,\cdot\rangle_1)$ which is one-to-to. Suppose that $w\in H_1$. Since $T:H_1\to H$ is an isomorphism, there is an element $u\in H_1$such that $Tu=w$. But part (b) of Lemma 1 now shows that $u\in D(S)=H_2$ and so $T(H_2)=H_1$. Thus $T:H_2\to H_1$ is onto and the first inequality in part (a) of Lemma 1 completes the proof that $T:H_2\to H_1$ is a homeomorphism.

(c) For $u\in H_1$, we have that $T^{-1}u\in H_2$ by part (b) and so $T^{-1}ST^{-1}u\in H_1$ with

$$\left\|T^{-1}ST^{-1}u\right\|_1=\left\|ST^{-1}u\right\|\leq\left\|T^{-1}u\right\|_2\leq\|u\|_1$$

by (10) and (11). Thus $T^{-1}ST^{-1}$ is a bounded operator from $(H_1, \langle\cdot,\cdot\rangle_1)$ into $(H_1, \langle\cdot,\cdot\rangle_1)$. Finally for $u \in H_2$ and $v \in H_1$, we have

$$\begin{aligned} \left\langle T^{-1}ST^{-1}u, v\right\rangle_1 &= \left\langle ST^{-1}u, Tv\right\rangle \text{ by (10)} \\ &= \left\langle T^{-1}Su, Tv\right\rangle \text{ by Lemma 1(c)} \\ &= \langle Su, v\rangle \end{aligned}$$

since $T^{-1} : H \to H$ is a bounded self-adjoint operator.

(d) For $u \in H$, we have that $T^{-1}u \in H_1$ by part (a) of Lemma 1 and so $T^{-1}T^{-1}u \in H_2$ by part (b) with

$$\left\|T^{-1}T^{-1}u\right\|_2 \le \left\|T^{-1}u\right\|_1 = \|u\|$$

by (11) and (10). Thus $T^{-1}T^{-1}$ is a bounded operator from $(H, \langle\cdot,\cdot\rangle)$ into $(H_2, \langle\cdot,\cdot\rangle_2)$. Finally for $u \in H$ and $v \in H_1$, we have

$$\begin{aligned} \left\langle T^{-1}T^{-1}u, v\right\rangle_1 &= \left\langle T^{-1}u, Tv\right\rangle \text{ by (10)} \\ &= \langle u, v\rangle \end{aligned}$$

since $T^{-1} : H \to H$ is a bounded self-adjoint operator.

We now introduce the representation of S as a bounded self-adoint operator acting on $(H_1, \langle\cdot,\cdot\rangle_1)$.

Theorem 3 *Let $S : D(S) \subset H \to H$ be a self-adjoint operator on H with form space $(H_1, \langle\cdot,\cdot\rangle_1)$.*
(i) There is a unique bounded linear operator A from $(H_1, \langle\cdot,\cdot\rangle_1)$ into itself such that

$$\langle Au, v\rangle_1 = \langle Su, v\rangle \text{ for all } u \in H_2 \text{ and } v \in H_1. \tag{12}$$

Furthermore,

$$\langle Au, v\rangle_1 = \langle u, Av\rangle_1 = \int \lambda d\, \langle E(\lambda)u, v\rangle \tag{13}$$

for all $u, v \in H_1$ where $\{E(\lambda) : \lambda \in \mathbb{R}\}$ is the resolution of the identity associated with S.

(ii) There is a unique bounded linear operator L from $(H_1, \langle\cdot,\cdot\rangle_1)$ into itself such that

$$\langle Lu, v\rangle_1 = \langle u, v\rangle \text{ for all } u, v \in H_1. \tag{14}$$

Furthermore, $\langle Lu, v\rangle_1 = \langle u, Lv\rangle_1$ for all $u, v \in H_1$.

Let $T = (I + |S|)^{1/2} : D(T) = H_1 \subset H \to H$ be the operator introduced in Lemma 1.

(iii) Then $A = T^{-1}ST^{-1}$ and $L = T^{-1}T^{-1}$.

(iv)

$$\sigma(S) = \{\lambda \in \mathbb{R} : \text{ the bounded operator } A - \lambda L : H_1 \to H_1 \text{ is an isomorphism}\}$$

(v)

$$\sigma_e(S) = \{\lambda \in \mathbb{R} : \text{ the bounded operator } A - \lambda L : H_1 \to H_1 \text{ is not Fredholm}\}$$

Proof (i) Setting $A = T^{-1}ST^{-1}$, Corollary 2 shows that this operator is bounded from $(H_1, \langle\cdot,\cdot\rangle_1)$ into itself and has the property (12). The symmetry and uniqueness of A follow from the fact that $D(S)$ is a dense subset of H_1.

For all $\lambda \in \mathbb{R}$, $E(\lambda)$ and T^{-1} are bounded self-adjoint operators on H which commute. Hence

$$\left\langle E(\lambda)T^{-1}u, Tv\right\rangle = \left\langle T^{-1}E(\lambda)u, Tv\right\rangle = \langle E(\lambda)u, v\rangle$$

for all $u, v \in H_1$. Thus, by (10),

$$\begin{aligned}\langle Au, v\rangle_1 &= \left\langle T^{-1}ST^{-1}u, v\right\rangle_1 = \left\langle ST^{-1}u, Tv\right\rangle \\ &= \int \lambda d\left\langle E(\lambda)T^{-1}u, Tv\right\rangle = \int \lambda d\left\langle E(\lambda)u, v\right\rangle\end{aligned}$$

for all $u, v \in H_1$ since $T^{-1}u \in D(S)$.

(ii) Similar to (i).

(iv) and (v) For all $\lambda \in \mathbb{R}, A - \lambda L = T^{-1}(S - \lambda I)T^{-1} = U(S - \lambda I)V$ where $U = T^{-1} : H \to H_1$ and $V = T^{-1} : H_1 \to H_2$ are linear homeomorphisms and $S - \lambda I : H_2 \to H$ is a bounded linear operator. Hence

$$\ker(A - \lambda L) = V^{-1}\ker(S - \lambda I)$$

and

$$rge\,(A - \lambda L) = rge(S - \lambda I).$$

It follows that $\dim\ker(A-\lambda L)=\dim\ker(S-\lambda I)$ and that $rge\,(A-\lambda L)$ is a closed subspace of H_1 if and only if $rge(S-\lambda I)$ is a closed subspace of H. From these observations we se that $S-\lambda I : D(S)\to H$ is an isomorphism (respectively a Fredholm operator) if and only if $A-\lambda L : H_1\to H_1$ is an isomorphism (respectively a Fredholm operator).The statements (iv) and (v) now follow from the definitions of $\sigma(S)$ and $\sigma_e(S)$ given in Section 2.

Finally we relate A to the polar decomposition of S.

Lemma 4 *Let $S : D(S)\subset H\to H$ be a self-adjoint operator on H with form space $(H_1,\langle\cdot,\cdot\rangle_1)$ and let $A : H_1\to H_1$ be the operator introduced in Theorem 3. Set*

$$Ru=[I-P]\,u-Pu=u-2Pu \text{ for } u\in H \tag{15}$$

where $P=E(0)$ and $\{E(\lambda):\lambda\in\mathbb{R}\}$ is the resolution of the identity associated with S.
(i) $R^2=I$ and R is a self-adjoint isometric isomorphism of $(H,\langle\cdot,\cdot\rangle)$ onto itself.
(ii) R is also a self-adjoint isometric isomorphism of $(H_1,\langle\cdot,\cdot\rangle_1)$ onto itself.
(iii) $AR=RA$ and

$$\langle RAu,v\rangle_1=\langle Au,Rv\rangle_1=\int|\lambda|\,d\,\langle E(\lambda)u,v\rangle=\left\langle |S|^{1/2}u,|S|^{1/2}v\right\rangle \tag{16}$$

for all $u,v\in H_1$. Thus

$$\langle RAu,v\rangle_1=\langle ARu,v\rangle_1=\langle |S|\,u,v\rangle$$

for all $u\in D(|S|)=D(S)$ and $v\in H_1$.
(iv) For all $u\in D(S)=D(|S|)$,

$$Ru\in D(S) \text{ and } SRu=RSu=|S|\,u. \tag{17}$$

Remark It follows that $S=\widetilde{R}\,|S|$ is the usual polar decomposition of S (Chapter IV §3 of [2], for example) where

$$\widetilde{R}u=\begin{cases}0 \text{ if } u\in\ker S\\ Ru \text{ if } u\in[\ker S]^{\perp}\end{cases}$$

and $[\ker S]^{\perp}$ denotes the orthogonal complement of $\ker S$ in $(H,\langle\cdot,\cdot\rangle)$.

Proof (i) $R^2 = I - 4P + 4P^2 = I$ since $P^2 = P$ and, for any $u \in H$,

$$\|[I-P]u \pm Pu\|^2 = \|[I-P]u\|^2 + \|Pu\|^2 .$$

(ii) By (7), $Pu \in H_1$ for all $u \in H_1$ and so $RH_1 \subset H_1$. Furthermore by (10), for $u \in H_1$,

$$\begin{aligned}\|Ru\|_1^2 &= \int (1+|\lambda|) d\langle E(\lambda)Ru, Ru\rangle \\ &= \int (1+|\lambda|) d\langle E(\lambda)u, u\rangle = \|u\|_1^2\end{aligned}$$

since $\langle E(\lambda)Ru, Ru\rangle = \langle RE(\lambda)u, Ru\rangle = \langle E(\lambda)u, u\rangle$ by part (i). Hence

$$\langle Ru, v\rangle_1 = \left\langle R^2u, Rv\right\rangle_1 = \langle u, Rv\rangle_1 \text{ for all } u, v \in H_1.$$

(iii) For $u, v \in H_1$,

$$\langle Au, Rv\rangle_1 = \int \lambda d\langle E(\lambda)u, Rv\rangle$$

by (13) where

$$\langle E(\lambda)u, Rv\rangle = \begin{cases} -\langle E(\lambda)u, v\rangle & \text{if } \lambda \le 0 \\ \langle E(\lambda)u, v\rangle - 2\langle E(0)u, v\rangle & \text{if } \lambda > 0 \end{cases}$$

and so

$$\langle Au, Rv\rangle_1 = \int |\lambda|\, d\langle E(\lambda)u, v\rangle .$$

If $u \in D(S) = D(|S|)$ and $v \in H_1$,

$$\left\langle |S|^{1/2} u, |S|^{1/2} v\right\rangle = \langle |S|\, u, v\rangle = \int |\lambda|\, d\langle E(\lambda)u, v\rangle . \tag{18}$$

Since $D(|S|) = D(S)$ is a dense subspace of H_1, the continuity of $|S|^{1/2} : H_1 \to H$ and of $A, R : H_1 \to H_1$ implies that

$$\langle Au, Rv\rangle_1 = \left\langle |S|^{1/2} u, |S|^{1/2} v\right\rangle$$

for all $u, v \in H_1$. Hence $\langle Au, Rv\rangle_1 = \langle Av, Ru\rangle_1 = \langle v, ARu\rangle_1$ for all $u, v \in H_1$. But $\langle Au, Rv\rangle_1 = \langle RAu, R^2 v\rangle_1 = \langle RAu, v\rangle_1$ and so

$$\langle ARu, v\rangle_1 = \langle RAu, v\rangle_1 \text{ for all } u, v \in H_1,$$

showing that $AR = RA$.

(iv) By (7), $Pu \in D(S)$ and $SPu = PSu$ for all $u \in D(S)$. Hence $Ru \in D(S)$ and $SRu = RSu$ for all $u \in D(S)$. By part (iii),

$$\langle |S|\, u, v\rangle = \langle Au, Rv\rangle_1 = \langle RAu, v\rangle_1 = \langle ARu, v\rangle_1 = \langle SRu, v\rangle$$

for all $u \in D(S) = D(|S|)$ and $v \in H_1$.
Thus $|S|\, u = RSu = SRu$ for all $u \in D(S) = D(|S|)$ since H_1 is dense in H.

4 The quadratic form

Let $S : D(S) \subset H \to H$ be a self-adjoint operator on H with form space $(H_1, \langle\cdot,\cdot\rangle_1)$ and let A and $L : H_1 \to H_1$ be the operators introduced in Theorem 3. The set

$$M = \{u \in H_1 : \langle Lu, u\rangle_1 = 1\}$$

is a smooth manifold of codimension one in H_1 and the tangent space, $T_u(M)$, to M at u is given by

$$T_u(M) = \{v \in H_1 : \langle Lu, v\rangle_1 = 0\}.$$

(See Example 27.2 in (d).)

The **quadratic form**, $J : H_1 \to \mathbb{R}$, associated with S is defined by

$$J(u) = \langle Au, u\rangle_1 \text{ for all } u \in H_1.$$

Since $A : H_1 \to H_1$ is a bounded symmetric linear operator, $J \in C^\infty(H_1, \mathbb{R})$ and

$$J'(u)v = 2\,\langle Au, v\rangle_1 \text{ for all } u, v \in H_1.$$

For $u \in H_2, J(u) = \langle Su, u\rangle$ and J is the unique continuous extension of $\langle Su, u\rangle$ to H_1.

The restriction of J to the manifold M is denoted by j and will be referred to as the **Rayleigh quotient** for S since

$$\frac{\langle Su, u\rangle}{\langle u, u\rangle} = j(\frac{u}{\|u\|}) \text{ for all } u \in D(S)\setminus\{0\}.$$

Thus $j \in C^{\infty}(M, \mathbb{R})$ and

$$j'(u)v = 2\langle Au, v\rangle_1 \text{ for all } u \in M \text{ and } v \in T_u(M).$$

(See Example 27.3 of (d).)

Lemma 5 *Let $S : D(S) \subset H \to H$ be a self-adjoint operator on H with form space $(H_1, \langle\cdot,\cdot\rangle_1)$ and let j denote its Rayleigh quotient. For all $u \in M$ and $\lambda \in \mathbb{R}$,*

$$\|j'(u)\|_* \leq 2\|(A-\lambda L)u\|_1 \leq \|j'(u)\|_*(1+\|u\|_1) + 2|j(u)-\lambda|$$

where

$$\|j'(u)\|_* = \sup\left\{\frac{|j'(u)v|}{\|v\|_1} : v \in T_u(M) \text{ and } v \neq 0\right\}$$

denotes the norm on $T_u(M)^*$.

Proof For $u \in M$ and $v \in T_u(M)$ with $v \neq 0$,

$$\frac{|j'(u)v|}{\|v\|_1} = \frac{2|\langle(A-\lambda L)u, v\rangle_1|}{\|v\|_1} \leq 2\|(A-\lambda L)u\|_1$$

for all $\lambda \in \mathbb{R}$. Thus $\|j'(u)\|_* \leq 2\|(A-\lambda L)u\|_1$.

For $u \in M$, define $P_u : H_1 \to H_1$ by

$$P_u v : v - \langle Lu, v\rangle_1 u \text{ for all } v \in H_1.$$

Clearly $P_u v \in T_u(M)$ for all $u \in M$ and $v \in H_1$ and

$$\|P_u v\|_1 \leq \|v\|_1 + |\langle u, v\rangle|\,\|u\|_1 \leq (1+\|u\|_1)\|v\|_1$$

since $\|u\| = 1$ and $\|v\| \leq \|v\|_1$.

Consider $u \in M$ and $v \in H_1$. Then, for any $\lambda \in \mathbb{R}$,

$$
\begin{aligned}
&\langle (A-\lambda L)u, v\rangle_1 = \langle Au, P_u v + \langle Lu, v\rangle_1 u\rangle_1 - \lambda \langle Lu, v\rangle_1 \\
&= \langle Au, P_u v\rangle_1 + \langle Lu, v\rangle_1 \{\langle Au, u\rangle_1 - \lambda\} \\
&= \langle Au, P_u v\rangle_1 + \langle u, v\rangle \{j(u) - \lambda\}
\end{aligned}
$$

and so

$$
\begin{aligned}
|\langle (A-\lambda L)u, v\rangle_1| &\le |\langle Au, P_u v\rangle_1| + \|u\| \|v\| |j(u)-\lambda| \\
&\le \frac{1}{2} |j'(u) P_u v| + \|v\|_1 |j(u)-\lambda| \\
&\le \frac{1}{2} \|j'(u)\|_* \|P_u v\|_1 + \|v\|_1 |j(u)-\lambda| \\
&\le \left\{ \frac{1}{2} \|j'(u)\|_* (1 + \|u\|_1) + |j(u)-\lambda| \right\} \|v\|_1 .
\end{aligned}
$$

Hence

$$
\|(A-\lambda L)u\|_1 \le \frac{1}{2} \|j'(u)\|_* (1 + \|u\|_1) + |j(u)-\lambda| .
$$

5 The main results

Using the notions introduced in Sections 3 and 4 we can now state the results mentioned in the introduction in full generality.

Theorem 6 *Let $S : D(S) \subset H \to H$ be a self-adjoint operator on H with form space $(H_1, \langle \cdot, \cdot \rangle_1)$ and let J and $j = J\,|_M$ denote the associated quadratic form and Rayleigh quotient as defined in Section 3. Then,*

$$
\sigma(S) = \{\lambda \in \mathbb{R} : \text{ the functional } J_\lambda \text{ does not satisfy (P-S) on } H_1\}
$$

where $J_\lambda(u) = J(u) - \lambda \langle u, u\rangle = \langle (A-\lambda L)u, u\rangle_1$ for $u \in H_1$ and

$$
\sigma_e(S) = \{\lambda \in \mathbb{R} : \text{ the functional } j \text{ does not satisfy (P-S) at level } \lambda \text{ on } M\}
$$

where $\sigma(S)$ and $\sigma_e(S)$ denote the spectrum and essential spectrum of S as defined in Section 2.

Remark If $S : H \to H$ is a bounded self-adjoint operator, $H = H_1$ (up to equivalence of norms) and we obtain the results in the simple form stated in the introduction.

Proof Suppose first that $\lambda \in \sigma(S)$. Then $S - \lambda I : D(S) \to H$ is not an isomorphism. Since S is self-adjoint (and hence closed), it follows (see Theorem 5.2 of [9], for example) that there is a sequence $\{u_n\} \subset D(S)$ such that

$$\|u_n\| = 1 \text{ and } \|(S - \lambda I)u_n\| \to 0.$$

Set

$$\alpha_n = \begin{cases} \|(S-\lambda I)u_n\|^{-1/4} \text{ if } \|(S-\lambda I)u_n\| \neq 0 \\ n \text{ if } \|(S-\lambda I)u_n\| = 0. \end{cases}$$

and

$$v_n = \alpha_n T u_n$$

where $T = (I + |S|)^{1/2} : H_1 \to H$ is the operator introduced in Lemma 1. Then $\alpha_n \to \infty$ and

$$v_n \in H_1 \text{ with } \|v_n\|_1 = \alpha_n \|Tu_n\|_1 \geq \alpha_n \|u_n\| = \alpha_n$$

by Lemma 1(b). Hence the sequence $\{v_n\}$ has no subsequence which converges strongly in H_1.

But

$$J_\lambda(v_n) = \alpha_n^2 \langle (A - \lambda L)Tu_n, Tu_n \rangle_1 = \alpha_n^2 \left\langle (S - \lambda I)u_n, T^2 u_n \right\rangle$$

by (10) since $u_n \in D(S)$. Hence

$$|J_\lambda(v_n)| \leq \alpha_n^2 \|(S - \lambda I)u_n\| \left\|T^2 u_n\right\|$$

where

$$\left\|T^2 u_n\right\| = \|Tu_n\|_1 \leq \sqrt{2} \|u_n\|_2$$

by (10) and (11). Now the sequence $\{\|T^2 u_n\|\}$ is bounded since

$$\begin{aligned} \|u_n\|_2^2 &= \|u_n\|^2 + \|Su_n\|^2 \\ &\leq 1 + \{\|(S - \lambda I)u_n\| + |\lambda| \|u_n\|\}^2 \\ &\leq 1 + \{\|(S - \lambda I)u_n\| + |\lambda|\}^2 . \end{aligned}$$

On the other hand

$$\alpha_n^2 \|(S-\lambda I)u_n\| \to 0 \text{ as } n \to \infty$$

by the definition of α_n.
Thus $J_\lambda(v_n) \to 0$ as $n \to \infty$.

Also, for all $w \in H_1$,

$$J_\lambda'(v_n)w = 2\langle (A-\lambda L)v_n, w\rangle_1 = 2\alpha_n \langle (S-\lambda I)u_n, Tw\rangle$$

by (10) and so

$$\left|J_\lambda'(v_n)w\right| \leq 2\alpha_n \|(S-\lambda I)u_n\| \|Tw\| = 2\alpha_n \|(S-\lambda I)u_n\| \|w\|_1 .$$

Hence

$$\left\|J_\lambda'(v_n)\right\|_* \leq 2\alpha_n \|(S-\lambda I)u_n\|$$

where $\alpha_n \|(S-\lambda I)u_n\| \to 0$ as $n \to \infty$ by the definition of α_n.

This shows that J_λ does not satisfy the condition (P-S) at level 0 on H_1.

Conversely, suppose that $\lambda \in \rho(S)$ and choose any $c \in \mathbb{R}$. Consider a sequence $\{u_n\} \subset H_1$ such that $J_\lambda(u_n) \to c$ and $\|J_\lambda'(u_n)\|_* \to 0$.
It follows from Theorem 3(iv), that $A - \lambda L : H_1 \to H_1$ is an isomorphism and so the exists a constant $k > 0$ such that

$$\|(A-\lambda L)u\|_1 \geq k\|u\|_1 \text{ for all } u \in H_1.$$

But

$$\begin{aligned} \|J_\lambda'(u)\|_* &= \sup\left\{\frac{2\left|\langle (A-\lambda L)u, v\rangle_1\right|}{\|v\|_1} : v \in H_1 \text{ with } v \neq 0\right\} \\ &= 2\|(A-\lambda L)u\|_1 \text{ for all } u \in H_1. \end{aligned}$$

Thus $\|u_n\|_1 \to 0$ and J_λ satisfies the condition (P-S) at level c on H_1. (In fact, if $c \neq 0$ there is no sequence in H_1 such that $J_\lambda(u_n) \to c$ and $\|J_\lambda'(u_n)\|_* \to 0$.)

We now turn to the Rayleigh quotient j and the essential spectrum of S. Suppose first that $\lambda \in \sigma_e(S)$. Then there exists a sequence (called a Weyl sequence, see Theorem 7.2 of [9], for example) $\{u_n\} \subset D(S) \cap M$ such that $\|(S-\lambda I)u_n\| \to 0$ and $u_n \rightharpoonup 0$ weakly in H as $n \to \infty$. Thus

$$j(u_n) - \lambda = \langle Au_n, u_n\rangle_1 - \lambda\langle Lu_n, u_n\rangle_1 = \langle (S-\lambda I)u_n, u_n\rangle ,$$

so

$$|j(u_n) - \lambda| \leq \|(S - \lambda I)u_n\| \, \|u_n\| = \|(S - \lambda I)u_n\|$$

and hence $j(u_n) \to \lambda$ as $n \to \infty$.
Furthermore, by Lemma 5,

$$\begin{aligned} \|j'(u_n)\|_* &\leq 2\,\|(A - \lambda L)u_n\|_1 = 2\,\left\|(S - \lambda I)T^{-1}u_n\right\| \text{ by (10)} \\ &= 2\,\left\|T^{-1}(S - \lambda I)u_n\right\| \text{ by Lemma 1, since } u_n \in D(S) \\ &\leq 2\,\left\|T^{-1}(S - \lambda I)u_n\right\|_1 = 2\,\|(S - \lambda I)u_n\| \text{ by (10)} \end{aligned}$$

and so $\|j'(u_n)\|_* \to 0$ as $n \to \infty$.
However, since M is a closed subset of H and $u_n \rightharpoonup 0 \notin M$ weakly in H, the sequence $\{u_n\}$ cannot have a subsequence which converges strongly in H_1 and hence in H. This shows that j does not satisfy the condition (P-S) at the level λ on M.

Conversely, let $\lambda \in \mathbb{R} \backslash \sigma_e(S)$ and consider a sequence $\{u_n\} \subset M$ such that $j(u_n) \to \lambda$ and $\|j'(u_n)\|_* \to 0$ as $n \to \infty$.
We begin by showing that the sequence $\{u_n\}$ is bounded in H_1. In fact for any $u \in M$,

$$\begin{aligned} \|u\|_1^2 &= \|u\|^2 + \left\langle |S|^{1/2} u, |S|^{1/2} u \right\rangle = 1 + \langle Au, Ru\rangle_1 \text{ by (16)} \\ &\leq 1 + \|Au\|_1 \, \|Ru\|_1 = 1 + \|Au\|_1 \, \|u\|_1 \text{ by Lemma 4} \\ &\leq 1 + \left\{ \frac{1}{2} \|j'(u)\|_* (1 + \|u\|_1) + |j(u)| \right\} \|u\|_1 \text{ by Lemma 5.} \end{aligned}$$

Thus

$$\left\{ 1 - \frac{1}{2} \|j'(u)\|_* \right\} \|u\|_1^2 \leq 1 + \left\{ \frac{1}{2} \|j'(u)\|_* + |j(u)| \right\} \|u\|_1$$

for all $u \in M$. Since $j(u_n) \to \lambda$ and $\|j'(u_n)\|_* \to 0$ as $n \to \infty$, if follows that $\{\|u_n\|_1\}$ is bounded.
Next we note that, since $A - \lambda L : H_1 \to H_1$ is a bounded Fredholm operator (of index zero) by Theorem 3(v), it follows from Theorem 3.15 in Chapter 1 of [2] that there exist an bounded linear operator $W : H_1 \to H_1$ and a compact linear operator $K : H_1 \to H_1$ such that

$$W(A - \lambda L) = I - K.$$

Thus

$$u_n = W(A - \lambda L)u_n + Ku_n \tag{19}$$

where

$$\|(A - \lambda L)u_n\|_1 \to 0 \text{ as } n \to \infty$$

since

$$\|(A - \lambda L)u_n\|_1 \leq \frac{1}{2}\|j'(u_n)\|_* (1 + \|u_n\|_1) + |J(u_n) - \lambda|$$

by Lemma 5 and the sequence $\{\|u_n\|_1\}$ is bounded.
But the boundedness of $\{\|u_n\|_1\}$ together with the compactness of K mean that there exist $z \in H_1$ and a subsequence $\{u_{n_i}\}$ such that $\|z - Ku_{n_i}\|_1 \to 0$ as $n_i \to \infty$. It follows from (19) that $u_{n_i} \to z$ in H_1 as $n_i \to \infty$. Since M is a closed subset of H_1 we have that $z \in M$.

Thus $\{u_n\}$ has a strongly convergent subsequence in M and we have shown that the functional j satisfies the condition (P-S) at the level λ on M.

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